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Design of Static H_∞ Linear Parameter Varying Controllers for Unmanned Aircraft

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I. Introduction

ONE of the main drawbacks in conventional gain-scheduling schemes in aircraft control is the lack of stability or performance guarantees over the entire operating envelope because controllers are designed at discrete operating points. Linear parameter varying (LPV) controllers offer a natural way of circumventing this problem by taking into account the dependence of aircraft dynamics over the scheduling variables during the synthesis process. Some of the applications include the design of LPV controllers for high performance aircraft with added assumptions on bounds on the rate of scheduling parameters [1,2], the design of the LPV controller for missiles and aircraft based on a solution of convex linear matrix inequalities (LMIs) [3,4] and μ synthesis [5,6], and LPV controllers using loop shaping for aircraft [7] and missiles [8]. In fact, one can say that with the flight testing of LPV controllers [7], this field of research has come of age. However, all of the previous references, except [8], deal with dynamic LPV controllers whose orders are generally high, consuming more time in implementation and testing. As a result, research into the design of static output feedback has assumed importance over the years. In [9], static output feedback controllers that minimize a quadratic cost function are presented in terms of modified Riccati equations and a rank minimization problem. A global minimum is, however, not guaranteed for the rank minimization problem. Sufficiency conditions for the existence of suboptimal controllers are presented in [10] and inversely coupled Lyapunov inequalities are solved using a min / max approach to obtain the controller. The min / max algorithm is modified in [11] to include extra scalar parameters to solve the Lyapunov inequalities and both necessary and sufficiency conditions are given for the existence of controllers. It is to be noted that both [10,11] rely on an iterative procedure for controller computation. In contrast to the iterative design procedures in [10,11], Benton and Smith in [12] present a noniterative method that guarantees a prescribed level of closed loop stability and output feedback gain. The design of static output feedback controllers that satisfy the H_∞ norm criterion is also formulated in terms of two sufficient LMI conditions [13], and an augmented Lagrangian approach is used in [14] to find an optimal static controller. The cone complementarity approach is used in [15] to find suboptimal static controllers. The main drawback of [13] is that the LMI conditions are problem dependent, while the optimal and suboptimal solutions proposed in [14,15], respectively, are local solutions. Recently, coupled LMI conditions were derived [16] to handle H_∞ loop-shaping problems with static output feedback, and an iterative solution is proposed [17] for solving coupled matrix design equations that lead to static H_∞ controllers. The results of [16] are extended in [8] to LPV systems using a loop-shaping technique. This work focuses on the design of static LPV controllers for an unmanned air vehicle (UAV) using an alternate approach to [8] for deriving static loop-shaping controllers for LPV plants. While [8] uses a coprime factorization approach, this Note uses a four-block formulation, thus decreasing the number of LMIs to be solved. It is to be noted that the final loop-shaping controller includes the weighting functions [18] and dynamic LPV loop-shaping controllers are more complicated and difficult to implement than static loop-shaping controllers.

A related issue in LPV controller design is the development of LPV models. In the case of UAVs, the dynamics are dependent on speed, altitude, angle of attack, and side-slip angle. Some of the methods employed in the development of LPV models include Jacobian linearization of nonlinear dynamics at different operating points throughout the flight envelope [19,20] and state transformation and function substitution (Marcos and Balas [20] present a detailed comparison of different LPV modeling techniques). In this Note, the longitudinal LPV model of the UAV is obtained through a series of approximations in the elements of the state-space matrices based on an understanding of flight dynamics, thus simplifying the model and consequently the controller design process. Because the UAV operates at low Reynolds number and a variation in height is negligible, LPV modeling and controller design use only speed as the scheduling variable. The simplified LPV model was also used in an earlier work [21] for the design of LPV controllers using μ synthesis.

This paper is organized as follows. In Sec. II, the development of the longitudinal LPV model of the UAV, design specifications, and the choice of LPV weights for the loop-shaping problem are presented. Section III presents conditions for the existence of static output LPV controllers, and Sec. IV presents the results of the application of the proposed design methodology on the UAV. Results from nonlinear simulation are also presented in Sec. IV. Finally, Sec. V concludes the Note with a summary of the results.

II. LPV Model and Problem Formulation

With a cruise speed of 22–72 m/s, the aircraft considered in this paper is a generic model representative of the class of UAVs that operate at low Reynolds number. The linear longitudinal models of the UAV are obtained from the local linearizations of the 6-degree of freedom nonlinear dynamic model and given by

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$$
\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} X_u(V) & X_w(V) & -g\cos(\theta_1(V)) & X_q(V) & 0 \\ Z_u(V) & Z_w(V) & Z_\theta(V) & Z_q(V) & 0 \\ 0 & 0 & 0 & 1 & 0 \\ M_u(V) & M_w(V) & M_\theta(V) & M_q(V) & 0 \\ 0 & -1 & V & 0 & 0 \end{bmatrix} \times \begin{bmatrix} u \\ w \\ \theta \\ q \\ h \end{bmatrix} + \begin{bmatrix} X_{\delta e}(V) & X_{\delta t}(V) \\ Z_{\delta e}(V) & Z_{\delta t}(V) \\ 0 & 0 \\ M_{\delta e}(V) & M_{\partial t}(V) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} \tag{1}
$$

$$
\begin{bmatrix} q \\ h \\ V_T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \cos(\alpha_1) & \sin(\alpha_1) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \\ q \\ h \end{bmatrix} \tag{2}
$$

The perturbed states in Eqs. (1) and (2) correspond to velocity u along the x axis of the body-axes coordinate system, velocity w along the z axis of the body-axes coordinate system, pitch attitude θ, pitch rate q, and height h. δ_e is the elevator input and δ_t is the throttle input. The measurements used for feedback are pitch rate, height, and total velocity V_T. Trim velocity V is the independent parameter defined as the velocity at which the aircraft is in equilibrium. θ_1 is the trim pitch angle and α_1 is the trim angle of attack. Note that for straight and level flight, $\theta_1 = \alpha_1$. The aim of the longitudinal controller design is to track the height and velocity reference signals. It is worthwhile to note that the dynamics of the model is independent of height and, hence, density. Although such an assumption can lead to large inaccuracies in the modeling of full-scale aircraft, it is perfectly valid for UAVs that operate in the low Reynolds number region, because the altitude does not vary significantly during cruise. The polynomial dependence of the dimensional derivatives on velocity in Eq. (1) is found through least-squares curve fitting. Although the derivatives vary in both linear and quadratic fashions as functions of velocity, only the linear dependence is used to make the problem simple and tractable. Also, only the parameter dependence of dimensional derivatives that most influence the dynamics of the UAV is retained in the final LPV model. This is achieved by varying each coefficient in turn over the entire range while fixing all other coefficients in Eqs. (1) and (2) at their average values. It is thus determined that the coefficients $X_q, Z_\theta, Z_q, M_u, Z_{\delta e}$, and $M_{\delta e}$ are the most significant ones in the sense that any variation in these coefficients would introduce large changes in the UAV response in terms of damping, natural frequency, and response amplitudes.

For purposes of confirming the veracity of the LPV model, longitudinal models at three representative flight speeds are compared. Figures 1–3 show the frequency responses of the linearized plant model and the LPV model at 22, 47, and 72 m/s, where the measured outputs are the total velocity, height, and pitch rate. The frequency responses at 22 and 72 m/s show a marked difference around the phugoid frequency and short-period frequency whereas the difference is negligible at the flight speed of 47 m/s. This is again expected, since 47 represents the midpoint of the cruise speed range, and average values are chosen for most of the stability and control derivatives. Any variation in the dynamic characteristics can be treated as uncertainties against which the controller should be robust.

The controller design problem considered in this paper is formulated as follows. Given the longitudinal LPV model $G(\rho)$ in Fig. 4a and loop-shaping weights $W_1(\rho)$ and $W_2(\rho)$, find a static LPV controller $K(\rho)$ such that the closed loop system satisfies the condition $\|T_{zw}\|_\infty < \gamma$ for a given γ, where T_{zw} is the transfer function matrix from the disturbances $[w_1 \ w_2]'$ to the outputs $[z_1 \ z_2]'$ [8]. To lend a physical interpretation to the disturbance and output

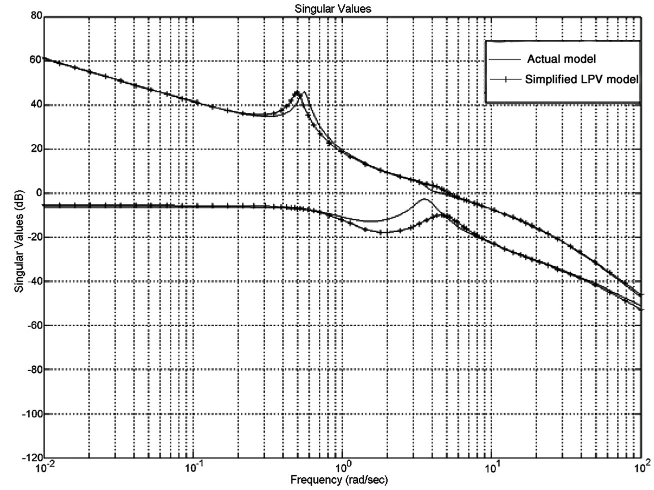

Fig. 1 Frequency responses of the actual model and simple LPV model at 22 m/s.

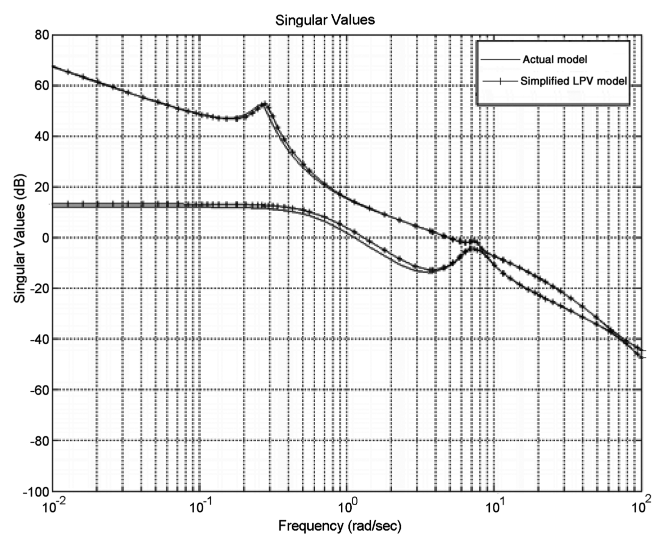

Fig. 2 Frequency responses of actual model and LPV model at 47 m/s.

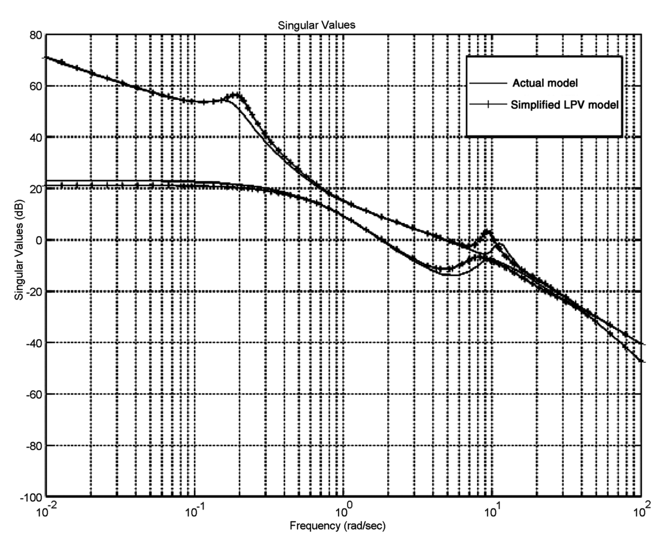

Fig. 3 Frequency responses of actual model and LPV model at 72 m/s.

signals, the loop-shaping problem in Fig. 4a is recast into a conventional closed loop configuration as shown in Fig. 4b, where K_l is given as $K_l = W_1 K W_2$. w_1 and w_2 are reference and plant input disturbance, respectively, while z_1 and z_2 are the weighted error and

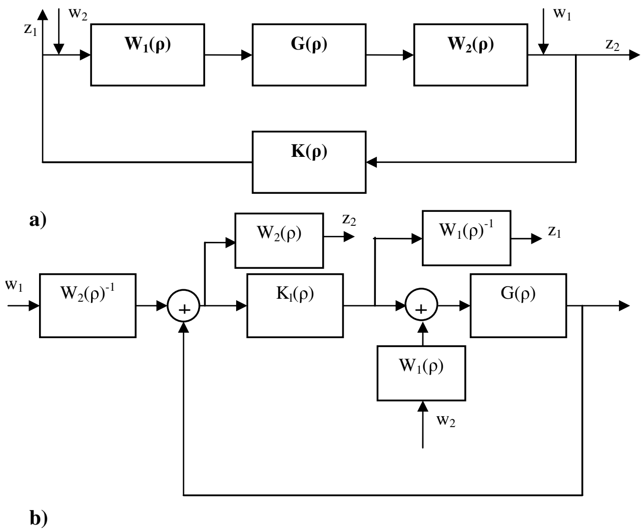

Fig. 4 a) Closed loop system with loop-shaping weights W_1 and W_2 and controller K and b) equivalent closed loop system with physical definition of signals, with $K_l = W_1 K W_2$.

control signals, respectively. The plant in Fig. 4a is augmented with strictly proper sensors and actuators at the outputs and inputs to collect all variations in the plant model in the A matrix of the plant.

In the absence in this study of any guidelines on performance of the UAVs, the design specifications are based on manned aircraft. The objective of controller design is to achieve good tracking of height and total velocity reference signals with fast and well-damped responses.

The specifications on the singular values of $W_2 G W_1$ include high values of $\underline{\sigma}(G_s)$ at low frequencies and low values of $\bar{\sigma}(G_s)$ at high frequencies. Also, a crossover frequency of 20–30 rad/s is chosen for fast responses

$$W_1(V) = \begin{bmatrix} \frac{(1.059V^2 - 131.44V + 4126.8)(0.01s+1)}{(s+0.0001)} & 0 \\ 0 & \frac{50(0.01s+1)}{(s+0.0001)} \end{bmatrix} \tag{3}$$

$$W_2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \tag{4}$$

The two diagonal elements in W_1 correspond to the two inputs of G, and the three diagonal elements of W_2 correspond to the three outputs, namely, pitch rate, height, and total velocity of G. The near-integral terms in W_1 are required to achieve near zero steady-state error for step demands on height and total velocity. Figure 5 shows the open-loop ($W_2 G W_1$) frequency responses at speeds of 22, 37, 42, 57, and 72 m/s.

III. Existence Conditions for Static LPV Controllers

Let the state-space realization of the shaped plant $G_s(\rho) = W_1(\rho) G(\rho) W_2(\rho)$ be given by

$$G_s := \begin{bmatrix} A(\rho) & B \\ C & D \end{bmatrix}$$

The time-varying parameter ρ ranges over the polytope

$$\Theta = \left\{ \sum_{i=1}^{r} \alpha_i \vartheta_i \colon \alpha_i \geq 0, \sum_{i=1}^{r} \alpha_i = 1 \right\}$$

with vertices $[\vartheta_1, \vartheta_2, \ldots, \vartheta_r]$. $A(\rho)$ is an affine function and ranges over the polytope of matrices with vertices $A(\vartheta_i)$. The problem is then to find an LPV controller $K(\theta)$ such that $\|T_{zw}\|_\infty < \gamma$ for a given $\gamma > 0$, where T_{zw} is the transfer function matrix from

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

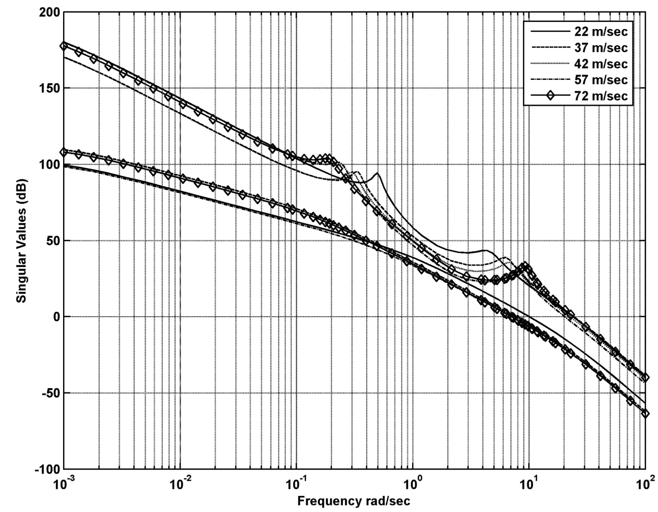

Fig. 5 Frequency response of $W_2 G W_1$.

$$\|T_{zw}\|_\infty = \left\| \begin{bmatrix} K(\rho)(I - G_s(\rho)K(\rho))^{-1} & K(\rho)(I - G_s(\rho)K(\rho))^{-1} G_s(\rho) \\ (I - G_s(\rho)K(\rho))^{-1} & (I - G_s(\rho)K(\rho))^{-1} G_s(\rho) \end{bmatrix} \right\|_\infty \tag{5}$$

Let $P(\rho)$ denote the strictly proper generalized plant of the feedback interconnection in Fig. 4. The strictly proper condition can be achieved by augmenting the plant with strictly proper actuators. The state-space realization of $P(\rho)$ is then

$$P(\rho) := \begin{bmatrix} A(\rho) & 0 & B & B \\ 0 & 0 & 0 & I \\ C & I & 0 & 0 \\ C & I & 0 & 0 \end{bmatrix} \tag{6}$$

Remark 1: Note that the variation of $G(\rho)$ is smooth for most practical applications and as a result, the variations of $W_1(\rho)$ and $W_2(\rho)$ are also smooth. In fact, it is possible to select the weighting functions such that the variations of the dynamics of the shaped plant are restricted to the A matrix alone.

Theorem 1: Given the generalized plant $P(\rho)$ in Eq. (6), the existence of a static loop-shaping controller $K(\rho)$ such that $\|T_{zw}\|_\infty < \gamma$, where T_{zw} is given in Eq. (5) and $\gamma > 0$, is guaranteed by the existence of a positive definite matrix R that satisfies the following inequalities:

$$\begin{pmatrix} A(\vartheta_i)R + RA(\vartheta_i)^T - \gamma BB^T & RC^T & 0 & B \\ CR & -\gamma I & I & 0 \\ 0 & I & -\gamma I & 0 \\ B^T & 0 & 0 & -\gamma I \end{pmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} RA(\vartheta_i)^T + A(\vartheta_i)R & B \\ B^T & -\gamma \end{bmatrix} < 0, \quad \text{for } i = 1, \ldots, r \tag{8}$$

Moreover, the static loop-shaping controller is given in terms of the vertex controllers K_i as

$$K(\rho) := \sum_{i=1}^{r} \alpha_i K_i$$

where

$$\rho = \sum_{i=1}^{r} \alpha_i \vartheta_i, \ \alpha_i \geq 0, \qquad \sum_{i=1}^{r} \alpha_i = 1$$

and K_i are found from the solution of the following system of LMIs:

$$\Psi_{Ri} + Q_R^T K_i P_R + P_R^T K_i Q_R < 0 \tag{9}$$

$$\Psi_{Ri} = \begin{pmatrix} A(\vartheta_i)R + RA(\vartheta_i)^T & R[0\;\; C^T] & [0\;\; B] \\ \begin{bmatrix} 0 \\ C \end{bmatrix} & -\gamma I & \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ B^T \end{bmatrix} & \begin{bmatrix} 0 & I \\ 0 & 0^T \end{bmatrix} & -\gamma I \end{pmatrix} \quad (10)$$

$$Q_R = [B^T \;\; I \;\; 0 \;\; 0], \qquad P_R = [C \;\; R \;\; 0 \;\; I \;\; 0] \quad (11)$$

Proof: The proof of this theorem follows closely with the proof for the existence of static loop-shaping controllers presented in [8,16]. From the standard results of [22], we arrive at the following conclusion. Given a generalized plant with the state-space realization

$$P_g := \begin{bmatrix} A_g(\rho) & B_{g1} & B_{g2} \\ C_{g1} & D_{g11} & D_{g12} \\ C_{g2} & D_{g21} & D_{g22} \end{bmatrix}$$

where $A_g(\rho)$ is stable for all ρ and varies over a polytope of matrices with vertices that are images of the vertices of the parameter polytope, a static LPV controller $K(\rho)$ exists if and only if there exist two symmetric positive definite matrices R and S such that

$$\begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A_g(\rho)R + RA_g(\rho)^T & RC_{g1}^T & B_{g1} \\ C_{g1}R & -\gamma I & D_{g11} \\ B_{g1}^T & D_{g11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} N_R & 0 \\ 0 & I \end{pmatrix} < 0 \quad (12)$$

$$\begin{pmatrix} N_S & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A_g(\rho)^T S + SA_g(\rho) & SB_{g1} & C_{g1}^T \\ B_{g1}^T S & -\gamma I & D_{g11}^T \\ C_{g1} & D_{11} & -\gamma I \end{pmatrix} \times \begin{pmatrix} N_S & 0 \\ 0 & I \end{pmatrix} < 0 \qquad R = S^{-1} \quad (13)$$

where N_R and N_s denote the bases of the null spaces of $(B_2^T,\; D_{12}^T)$ and $(C_2,\; D_{21})$, respectively. For the generalized plant $P(\rho)$ in Eq. (6), we have $(B_{g2}^T\; D_{g12}^T) = [B^T\; I\; 0]$ and $(C_{g2}^T\; D_{g21}^T) = [C\; I\; 0]$. Taking into account the size of the matrices and their ranks, the bases of the null space of $[B^T\;\; I\;\; 0]$ is chosen as

$$\begin{bmatrix} I & 0 \\ -B^T & 0 \\ 0 & I \end{bmatrix}$$

and the bases of the null space of $[C\;\; I\;\; 0]$ are chosen as

$$\begin{bmatrix} I & 0 \\ -C & 0 \\ 0 & I \end{bmatrix}.$$

Applying the LMI conditions (12) and (13) to the generalized plant $P(\theta)$, we have

$$\begin{pmatrix} A(\rho)R + RA(\rho)^T - \gamma BB^T & RC^T & 0 & B \\ CR & -\gamma I & I & 0 \\ 0 & I & -\gamma I & 0 \\ B^T & 0 & 0 & -\gamma I \end{pmatrix} < 0 \quad (14)$$

$$RA(\rho)^T + A(\rho)R - R\gamma C^T CR + B\gamma^{-1}B^T < 0 \quad (15)$$

Using the polytopic nature of the varying matrix $A(\rho)$, inequality (14) leads to Eq. (7) (see, for example, [23]). However, the presence of the quadratic term $R\gamma C^T CR$ removes any possibility of reducing Eq. (15) into a finite number of inequalities. This problem can be circumvented by requiring $RA(\rho)^T + A(\rho)R + BB^T\gamma^{-1} < 0$, which leads to inequality (8), with the additional constraint that $A(\theta)$ be at least stable for all ρ. Note that the removal of quadratic term $R\gamma C^T CR$ in Eq. (15) reduces the necessary and sufficient conditions into a sufficient condition. To retrieve the loop-shaping controller $K(\rho)$, note that the state-space realization of the closed loop transfer function from

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

in Fig. 4 is given by

$$G_{\mathrm{cl}} := \begin{bmatrix} A_{\mathrm{cl}}(\rho) & B_{\mathrm{cl}}(\rho) \\ C_{\mathrm{cl}}(\rho) & C_{\mathrm{cl}}(\rho) \end{bmatrix} = \begin{bmatrix} A(\rho) + BK(\rho)C & BK(\rho) & B \\ K(\rho)C & K(\rho) & 0 \\ C & I & 0 \end{bmatrix}$$

It is well known from the bounded real lemma (see, for example, [16]) that the closed loop system in Fig. 4 is stable with $\|T_{zw}\|_\infty < \gamma$ for a given $\gamma > 0$, if and only if there exists an $R > 0$, such that

$$\begin{bmatrix} A_{\mathrm{cl}}(\rho)^T R + RA_{\mathrm{cl}}(\rho) & RB_{\mathrm{cl}}(\rho) & C_{\mathrm{cl}}(\rho)^T \\ B_{\mathrm{cl}}(\rho)^T R & -\gamma I & D_{\mathrm{cl}}(\rho)^T \\ C_{\mathrm{cl}}(\rho) & -D_{\mathrm{cl}}(\rho) & -\gamma I \end{bmatrix} < 0$$

The above inequality can be transformed into the system of LMIs (9) with Ψ_{Ri}, Q_R, and P_R given by Eqs. (10) and (11). □

Remark 2: Though the constraint that $A(\rho)$ be at least stable for all ρ is conservative, it encompasses many practical design problems, including the UAV control problem presented here.

Remark 3: In [8], the static LPV loop-shaping problem is solved through the solution of two LMIs and an additional optimization problem. In contrast, the present approach only uses two LMIs.

IV. Design Results and Discussions

The vertex static controllers are obtained using the loop-shaping weights and generalized plant model in Eqs. (3), (4), and (6). The time-varying parameter in the problem is the total velocity V. Although W_1 varies quadratically with V [Eq. (3)], G varies in a linear fashion with V [Eq. (1)]. As a result the shaped plant W_1GW_2 varies in a cubic fashion with V, that is,

$$\left(\rho = \begin{bmatrix} V \\ V^2 \\ V^3 \end{bmatrix}\right)$$

To make the time-varying vector θ polytopic, the elements of θ are assumed to be independent. Thus,

$$\rho = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

with $22 \le V_1 \le 72$, $22^2 \le V_2 \le 72^2$, and $22^3 \le V_3 \le 72^3$. The parameter ρ then varies in a polytope Θ of vertices ϑ_i, $i = 1, \ldots, 8$, with

$$\vartheta_1 = \begin{bmatrix} 22 \\ 22^2 \\ 22^3 \end{bmatrix}, \quad \vartheta_2 = \begin{bmatrix} 22 \\ 22^2 \\ 72^3 \end{bmatrix}, \quad \vartheta_3 = \begin{bmatrix} 22 \\ 72^2 \\ 22^3 \end{bmatrix}$$

$$\vartheta_4 = \begin{bmatrix} 22 \\ 72^2 \\ 72^3 \end{bmatrix}, \quad \vartheta_5 = \begin{bmatrix} 72 \\ 72^2 \\ 22^3 \end{bmatrix}, \quad \vartheta_6 = \begin{bmatrix} 72 \\ 22^2 \\ 72^3 \end{bmatrix}$$

$$\vartheta_7 = \begin{bmatrix} 72 \\ 72^2 \\ 72^3 \end{bmatrix}, \quad \vartheta_8 = \begin{bmatrix} 72 \\ 22^2 \\ 22^3 \end{bmatrix}$$

The static LPV controller $K(\rho)$ is given by

$$K(\rho) = \sum_1^8 \alpha_i K_i$$

where K_i represent the vertex controllers. α_i is the solution to the complex decomposition problem

$$\rho = \sum_{1}^{8} \alpha_i \vartheta_i$$

and is obtained as follows:

$$\alpha_1 = \left(\frac{72 - V_1}{50}\right)\left(\frac{72^2 - V_2}{4700}\right)\left(\frac{72^3 - V_3}{362,600}\right)$$

$$\alpha_2 = \left(\frac{72 - V_1}{50}\right)\left(\frac{72^2 - V_2}{4700}\right)\left(\frac{V_3 - 22^3}{362,600}\right)$$

$$\alpha_3 = \left(\frac{72 - V_1}{50}\right)\left(\frac{V_2 - 22^2}{4700}\right)\left(\frac{72^3 - V_3}{362,600}\right)$$

$$\alpha_4 = \left(\frac{72 - V_1}{50}\right)\left(\frac{V_2 - 22^2}{4700}\right)\left(\frac{V_3 - 22^3}{362,600}\right)$$

$$\alpha_5 = \left(\frac{V_1 - 22}{50}\right)\left(\frac{V_2 - 22^2}{4700}\right)\left(\frac{72^3 - V_3}{362,600}\right)$$

$$\alpha_6 = \left(\frac{V_1 - 22}{50}\right)\left(\frac{72^2 - V_2}{4700}\right)\left(\frac{V_3 - 22^3}{362,600}\right)$$

$$\alpha_7 = \left(\frac{V_1 - 22}{50}\right)\left(\frac{V_2 - 22^2}{4700}\right)\left(\frac{V_3 - 22^3}{362,600}\right)$$

$$\alpha_8 = \left(\frac{V_1 - 22}{50}\right)\left(\frac{72^2 - V_2}{4700}\right)\left(\frac{72^3 - V_3}{362,600}\right)$$

The value of γ is found to be 2.8. Substituting for $V_1 = V$, $V_2 = V^2$, and $V_3 = V^3$, and introducing the variable $\bar{V} = V/50$ ($\bar{V}$ varies from 0.44 to 1.44 as V changes from 22 to 72 m/s) the controller structure is determined to be

$$K(\bar{V}) = \begin{bmatrix} K_{11}(\bar{V}) & K_{12}(\bar{V}) & K_{13}(\bar{V}) \\ K_{21}(\bar{V}) & K_{22}(\bar{V}) & K_{23}(\bar{V}) \end{bmatrix}$$

where

$$K_{11}(\bar{V}) = \mathbf{-2.4857}\bar{V}^6 + \mathbf{13.114}\bar{V}^5 \mathbf{-28.94}\bar{V}^4 + \mathbf{33.11}\bar{V}^3 \mathbf{-19.146}\bar{V}^2 + \mathbf{6.5589}\bar{V} \mathbf{-0.8214}$$

$$K_{12}(\bar{V}) = \mathbf{-2.1438}\bar{V}^6 + 11.891\bar{V}^5 \mathbf{-26.179}\bar{V}^4 + 29.764\bar{V}^3 \mathbf{-19.875}\bar{V}^2 + 6.365\bar{V} \mathbf{-0.8789}$$

$$K_{13}(\bar{V}) = \mathbf{-0.1631}\bar{V}^6 + 0.9236\bar{V}^5 \mathbf{-2.0387}\bar{V}^4 + 2.418\bar{V}^3 \mathbf{-1.587}\bar{V}^2 + 0.5169\bar{V} \mathbf{-0.0879}$$

$$K_{21}(\bar{V}) = 39.641\bar{V}^6 \mathbf{-213.65}\bar{V}^5 + 467.1\bar{V}^4 \mathbf{-530.84}\bar{V}^3 + \mathbf{329.9}\bar{V}^2 \mathbf{-107.3}\bar{V} + 14.2$$

$$K_{22}(\bar{V}) = 2.183\bar{V}^6 \mathbf{-12.29}\bar{V}^5 + 27.903\bar{V}^4 \mathbf{-33.159}\bar{V}^3 + \mathbf{21.479}\bar{V}^2 \mathbf{-7.235}\bar{V} + \mathbf{0.9539}$$

$$K_{23}(\bar{V}) = 0.0048\bar{V}^6 \mathbf{-0.0693}\bar{V}^5 + 0.2396\bar{V}^4 \mathbf{-0.3785}\bar{V}^3 + \mathbf{0.2481}\bar{V}^2 \mathbf{-0.08379}\bar{V} + \mathbf{0.017903}$$

It is to be noted that the final controller is obtained by augmenting the static controller $K(\bar{V})$ with the loop-shaping weights $W_1(\bar{V})$ and $W_2(\bar{V})$. In other words, the controller $K(V)$ is static in the sense that the controller parameters are not explicitly dependent on time, but the controller does depend on the varying parameter V, thus indirectly changing with time. The singular value plot of output sensitivity is shown in Fig. 6 for the representative velocities of 22, 47, and 72 m/s. As can be seen, the closed loop system exhibits good disturbance rejection properties at low frequency for the controlled channels, that is, height and total velocity at all trim speeds.

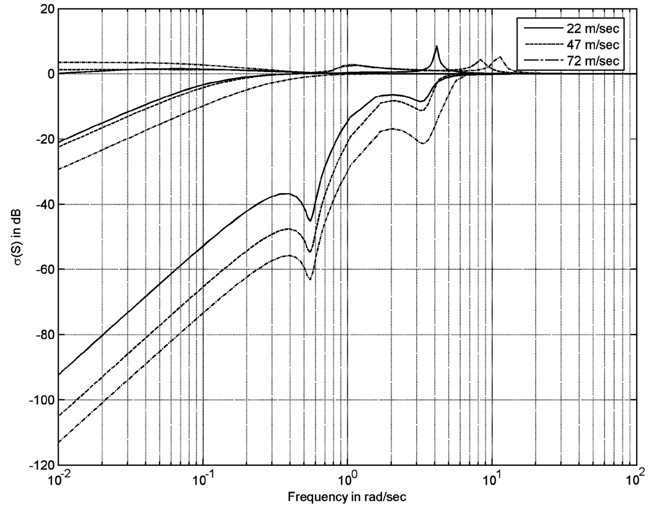

Fig. 6 Singular values of output sensitivity S_o.

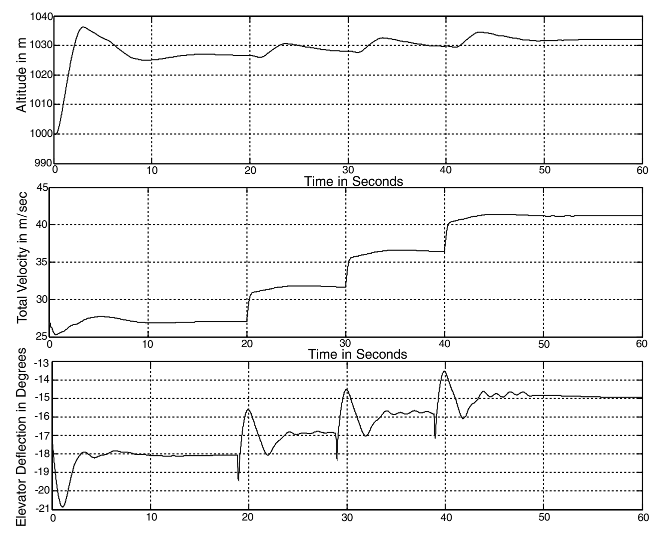

Fig. 7 Closed loop responses of total velocity and elevator deflection for 10 m step demand on height channel.

Remark 4: The assumption that V, V^2, and V^3 represent independent parameters leads to a conservative design. However, this also simplifies the computation of the controller using the method in Sec. V.

Nonlinear simulations: The efficacy of the controller is tested in a nonlinear setup with complete UAV dynamics replacing the LPV model. The aircraft is first trimmed at an operating speed of 27 m/s at an altitude of 1000 m. A step reference signal of 30 m is applied to the altitude channel at the start of the simulation. To introduce changes in the scheduling variable, the operating speed is increased in steps of 5 m/s at 20, 30, and 40 s. Though such sudden changes in velocity are impractical, they help us understand the behavior of the closed loop system in extreme conditions. The controller exhibits good robustness as can be seen from the height response in Fig. 7. The steady-state error of 2 m is acceptable considering the fact that the controller is designed for an LPV model that is an approximation of the linear model. Figure 7 also shows the elevator deflection and corrective action is initiated whenever there is a change in the scheduling parameter.

V. Conclusions

The problem of designing longitudinal LPV controllers for UAVs is dealt with using a loop-shaping approach. To simplify the implementation of such controllers, a new approach is presented for designing static output feedback loop-shaping LPV controllers. The present methodology transforms the loop-shaping problem into a four-block problem and the resultant controller is then expressed in terms of the solution of two simultaneous LMIs. To reduce the LMIs into a finite number of inequalities, three assumptions are made: all parameter dependence is restricted to the A matrix; A is at least stable and the scheduling parameters V (velocity), V^2 and V^3 are independent (the time-varying vector is polytpoic). The first assumption can be achieved by using parameter independent actuators and sensors, whereas the second assumption reduces sufficient and necessary conditions into sufficiency only conditions. It is also assumed that V can vary infinitely fast. Nonetheless, the method proposed in this paper is applicable to many design problems encountered in practice for fixed-wing aircraft. Information on the rate of parameter variation can potentially be used to achieve less conservative designs and is the topic for future research. Nonlinear simulations show that the static LPV controller performs well even in the presence of extreme changes in the scheduling parameter. The work is part of a larger program of research, which will design and build a UAV for testing new technologies such as fluidic thrust vectoring, circulation control, and the LPV approach described in this Note.

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